Semi Fineale 1

1 **Integral**: $\int \frac{\ln(\sin x)}{\cos^2 x}\,dx$

Solution: Using Integration by Parts we get

$$\begin{aligned}\int \frac{\ln(\sin x)}{\cos^2 x}\,dx &= \tan(x)\ln(\sin(x)) - \int \tan(x)\frac{\cos(x)}{\sin(x)} \\ &= \tan(x)\ln(\sin(x)) - x.\end{aligned}$$

2 **Integral**: $\int_0^{\pi} \frac{1}{2024-\cos x}\,dx$

Solution: Substitute $t = \tan\left(\frac{x}{2}\right)$ with $\frac{dx}{dt} = \frac{2}{1+t^2}$ and $\cos(x) = \frac{1-t^2}{1+t^2}$ then the Integral is

$$\begin{aligned}\int_0^{\infty} \frac{2}{2024(1+t^2)-(1-t^2)}\,dt &= \int_0^{\infty} \frac{2}{2025}\frac{1}{t^2+\left(\sqrt{\frac{2023}{2025}}\right)^2}\,dt \\ &= \frac{\pi}{\sqrt{2025\cdot 2023}}.\end{aligned}$$

3 **Integral**: $\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin(2x)}\,dx$

Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin(2x)}\,dx &= \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x-\sin x}{1+(\sin x-\cos x)^2}\,dx \\ &= \int_{-1}^{1} \frac{1}{1+z^2}\,dz - \int_0^{\frac{\pi}{2}} \frac{\sin x}{2-\sin(2x)}\,dx \\ &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin(2x)}\,dx\end{aligned}$$

by substituting x with $\frac{\pi}{2}-x$. So the Integral equals $\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$.

4 **Integral**: $\int_0^{\frac{\pi}{6}} \frac{1}{\cos x}\,dx$

Solution:

$$\int_0^{\frac{\pi}{6}} \frac{1}{\cos x}\,dx = \int_0^{\frac{\pi}{6}} \frac{\cos x}{\cos^2 x}\,dx$$

and with $t = \sin x$ this becomes

$$\int_0^{\frac{1}{2}} \frac{1}{1-t^2}\,dt = \frac{\ln 3}{2}.$$

5 **Integral**: $\int \frac{x}{\cos^2 x}\,dx$

Solution: Do Integration by parts to get

$$\tan(x)x - \int \tan(x)\,dx = \tan(x)x + \ln(\cos(x)).$$

6 **Integral** (Tiebreaker): $\int \frac{xe^{x^2}}{e^{2x^2} + 2e^{x^2} - 3} \, dx$

Solution: Substitute $z = e^{x^2}$ then the Integral becomes

$$\int \frac{1}{2} \frac{1}{z^2 + 2z - 3} \, dz = \frac{1}{8} \ln \left(\frac{1 - z}{3 + z} \right).$$

Semi Finale 2

1 **Integral**: $\int \sin(x)\cos(x)\ln\left(\sin^4(x)-\cos^4(x)\right)dx$

Solution:

$$\int \sin(x)\cos(x)\ln\left(\sin^4(x)-\cos^4(x)\right)dx = \int \frac{1}{4}\cdot 4\sin(x)\cos(x)\ln(1-2\cos^2(x))\,dx.$$

Substituting $t = 1-2\cos^2(x)$ gives

$$\begin{aligned}\int \frac{1}{4}\ln(t)\,dt &= \frac{1}{4}t(\ln(t)-1)\\ &= \frac{1}{4}(1-2\cos^2(x))(\ln(1-2\cos^2(x))-1).\end{aligned}$$

2 **Integral**: $\int_0^\infty \int_0^\infty e^{-3x}\cos(xy)\,dx\,dy$

Solution: $\cos(x) = \mathrm{Re}(e^{ixy})$ so the integral equals

$$\begin{aligned}\int_0^\infty \mathrm{Re}\int_0^\infty e^{-3x}e^{ixy}\,dx\,dy &= \int_0^\infty \mathrm{Re}\,\frac{1}{3-iy}\,dy\\ &= \frac{\pi}{2}.\end{aligned}$$

3 **Integral**: $\int_0^{\frac{\pi}{3}} \frac{1}{\cos^3(x)}\,dx$

Solution:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \frac{\cos x}{\cos^4 x}\,dx &= \int_0^{\frac{1}{2}} \frac{1}{(1-t^2)^2}\,dt\\ &= \int_0^{\frac{1}{2}} \frac{1}{4(1+t)} + \frac{1}{4(1+t)^2} - \frac{1}{4(1-t)} + \frac{1}{4(1-t)^2}\,dt\\ &= \frac{1}{3} + \frac{\ln(3)}{4}\end{aligned}$$

using the Substitution $t = \sin x$.

4 **Integral**: $\int_{\frac{1}{e}}^{e} \frac{1+\ln(x)}{x}\cdot\frac{x^2+x+1}{x^2+1}\,dx$

Solution: Substituting $\frac{1}{x}$ yields

$$\int_{\frac{1}{e}}^{e} \frac{\ln(x)}{x}\cdot\frac{x^2+x+1}{x^2+1}\,dx = \int_{\frac{1}{e}}^{e} \frac{-\ln(x)}{x}\cdot\frac{x^2+x+1}{x^2+1}\,dx$$

so the Integral becomes

$$\begin{aligned}\int_{\frac{1}{e}}^{e} \frac{1}{x}\cdot\frac{x^2+x+1}{x^2+1}\,dx &= \int_{\frac{1}{e}}^{e} \frac{1}{x} + \frac{1}{x^2+1}\,dx\\ &= 2+\arctan(e)-\arctan(1/e).\end{aligned}$$

5 **Integral**: $\int_0^{\frac{\sqrt{2}}{2}} x\arccos(x)\,dx$

Solution: Substituting $x = \cos(t)$ using $2\sin(t)\cos(t) = \sin(2t)$ and using integration by Parts we get

$$\begin{aligned}\int_0^{\frac{\sqrt{2}}{2}} x \arccos(x)\, dx &= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -t \sin(t)\cos(t)\, dt \\ &= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -\frac{1}{2} t \sin(2t)\, dt \\ &= \frac{1}{4} t \cos(2t)|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{1}{2}\cos(2t)\, dt.\end{aligned}$$

6 **Integral** (Tiebreak): $\int_0^\infty x^{\frac{-x^2}{2\ln(x)}}\, dx$

Solution:

$$\begin{aligned}\int_0^\infty x^{\frac{-x^2}{2\ln(x)}}\, dx &= \int_0^\infty e^{\frac{-x^2}{2}}\, dx \\ &= \sqrt{\frac{\pi}{2}}.\end{aligned}$$

Determination of the third place

[1] **Integral**: $\int x \arctan^2(x)\,dx$

Solution:

$$\begin{aligned}\int x \arctan^2(x)\,dx =& \frac{x^2}{2}\arctan^2(x) - \int x^2 \frac{\arctan(x)}{1+x^2}\,dx \\ =& \frac{x^2}{2}\arctan^2(x) - \left(\int \arctan(x)\,dx - \int \frac{\arctan(x)}{1+x^2}\,dx\right) \\ =& \frac{x^2}{2}\arctan^2(x) - \left(\arctan(x)x - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}\arctan^2(x)\right) \\ =& \frac{x^2+1}{2}\arctan^2(x) - \arctan(x)x + \frac{1}{2}\ln(1+x^2)\end{aligned}$$

Zu $\int \arctan(x)\,dx$:

$$\int \arctan(x)\,dx = \int 1\cdot\arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx = \arctan(x)x - \frac{1}{2}\ln(1+x^2).$$

[2] **Integral**: $\int_0^\infty (e^{-\frac{2}{x^2}} - e^{-\frac{8}{x^2}})\,dx$

Solution: The substitution $t = 1/x^2$ yields

$$\begin{aligned}\int_0^\infty (e^{-\frac{2}{x^2}} - e^{-\frac{8}{x^2}})\,dx &= \frac{1}{2}\int_0^\infty \frac{e^{-2t}-e^{-8t}}{t\sqrt{t}}\,dt \\ &= \frac{1}{2}\int_0^\infty \frac{1}{\sqrt{t}}\int_2^8 e^{-tu}\,du\,dt \\ &= \frac{1}{2}\int_2^8 \int_0^\infty \frac{e^{-tu}}{\sqrt{t}}\,dt\,du.\end{aligned}$$

By further substituting $tu = v^2$ we get

$$\begin{aligned}\frac{1}{2}\int_2^8\int_0^\infty \frac{2}{\sqrt{u}}e^{-v^2}\,dv\,du &= \frac{1}{2}\int_2^8 \frac{2}{\sqrt{u}}\int_0^\infty e^{-v^2}\,dv \\ &= \frac{1}{2}\sqrt{\pi}\int_2^8 \frac{1}{\sqrt{u}}\,du \\ &= \frac{\sqrt{\pi}}{2}2\sqrt{2} \\ &= \sqrt{2\pi}.\end{aligned}$$

[3] **Integral**: $\int_1^\infty \frac{\ln x}{x(x-1)}\,dx$

Solution: We use the substitution $x = e^t$ to get

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x(x-1)}\,dx &= \int_0^\infty \frac{t}{e^t-1}\,dt\\ &= \int_0^\infty \frac{te^{-t}}{1-e^{-t}}\,dt\\ &= \int_0^\infty te^{-t}\sum_{n=0}^\infty e^{-nt}\,dt\\ &= \sum_{n=0}^\infty \int_0^\infty te^{-(n+1)t}\,dt\end{aligned}$$

where we now substitute $u = (n+1)t$ to get

$$\begin{aligned}\sum_{n=0}^\infty \int_0^\infty \frac{u}{(n+1)^2}e^{-u}\,du &= \sum_{n=0}^\infty \frac{1}{(n+1)^2}\int_0^\infty ue^{-u}\,du\\ &= \zeta(2)\Gamma(2) = \frac{\pi^2}{6}.\end{aligned}$$

4 **Integral**: $\int_0^\infty \frac{\cos(184x)}{x^2+121}\,dx$

Solution: Consider

$$\begin{aligned}I(t) &= \int_0^\infty \frac{\cos(tx)}{x^2+11^2}\,dx\\ I'(t) &= \int_0^\infty \frac{-x\sin(tx)}{x^2+11^2}\,dx.\end{aligned}$$

Differentiating under the integral now gives the wrong result due to convergence problems, hence we use the following transformation:

$$\begin{aligned}I'(t) &= \int_0^\infty \frac{-x\sin(tx)}{x^2+11^2}\,dx\\ &= \int_0^\infty \frac{\sin(tx)}{x}\left(-1+\frac{11^2}{x^2+11^2}\right)dx.\end{aligned}$$

Using

$$\begin{aligned}\int_0^\infty \frac{\sin(tx)}{x}\,d &= \int_0^\infty \frac{\sin(tx)}{tx}t\,dx\\ &= \int_0^\infty \frac{\sin(u)}{u}\,du\\ &= \frac{\pi}{2}\end{aligned}$$

it follows that

$$\begin{aligned}I'(t) &= \int_0^\infty \frac{11^2}{x^2+11^2}\frac{\sin(tx)}{x}\,dx - \frac{\pi}{2}\\ I''(t) &= \int_0^\infty \frac{11^2\cos(tx)}{x^2+11^2}\,dx = 11^2 I(t).\end{aligned}$$

Thus the solutions are functions of the form

$$C_1e^{11t} + C_2e^{-11t}$$

where

$$\begin{aligned} C_1 + C_2 = I(0) &= \int_0^\infty \frac{1}{x^2+11^2}\,dx \\ &= \frac{1}{11}\arctan(x)\Big|_0^\infty \\ &= \frac{\pi}{2\cdot 11} \end{aligned}$$

and

$$11(C_1 - C_2) = I'(0) = -\frac{\pi}{2}.$$

Therefore $C_1 = 0$, $C_2 = \dfrac{\pi}{2\cdot 11}$ and

$$I(t) = \frac{\pi}{2\cdot 11}e^{-11t}$$

so overall

$$\int_0^\infty \frac{\cos(184x)}{x^2+11^2}\,dx = I(184) = \frac{\pi}{22}e^{-2024}.$$

5 **Integral**: $\displaystyle\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$

Solution:

$$\begin{aligned} \int_0^1 \frac{x^2}{(1+x^2)^2}\,dx &= \int_0^1 -\frac{1}{2}x\frac{-2x}{(1+x^2)^2}\,dx \\ &= -\frac{1}{2}x\frac{1}{1+x^2}\Big|_0^1 - \int_0^1 -\frac{1}{2}\frac{1}{1+x^2}\,dx \\ &= -\frac{1}{4} + \frac{1}{2}\arctan(x)\Big|_0^1 \\ &= -\frac{1}{4} + \frac{\pi}{8} \end{aligned}$$

using $\dfrac{d}{dx}\dfrac{1}{1+x^2} = \dfrac{-2x}{(1+x^2)^2}$.

6 **Integral**: $\displaystyle\int_0^{2\pi} \frac{1}{9\cos^2(t)+4\sin^2(t)}\,dt$

Solution: First consider

$$2\pi i = \int_D \frac{1}{z}dz$$

where D is an ellipse with half-axis lengths of 3 and 2. By parameterization we get

$$\begin{aligned} 2\pi i &= \int_0^{2\pi} \frac{-3\sin(t)+2i\cos(t)}{3\cos(t)+2i\sin(t)}\,dt \\ &= \int_0^{2\pi} \frac{(-3\sin(t)+2i\cos(t))(3\cos(t)-2i\sin(t))}{9\cos^2(t)+4\sin^2(t)}\,dt \\ &= \int_0^{2\pi} \frac{-5\cos(t)\sin(t)+6i}{9\cos^2(t)+4\sin^2(t)}\,dt. \end{aligned}$$

By comparing real and imaginary part we obtain

$$2\pi = 6\int_0^{2\pi} \frac{1}{9\cos^2(t) + 4\sin^2(t)}\,dt$$

which leads to

$$\int_0^{2\pi} \frac{1}{9\cos^2(t) + 4\sin^2(t)}\,dt = \frac{\pi}{3}.$$

Finale

1 **Integral**: $\int \frac{1}{x^{2024}+x}\,dx$

Solution:

$$\begin{aligned}\int \frac{1}{x^{2024}+x}\,dx &= \int \frac{1}{x} - \frac{x^{2022}}{x^{2023}+1}\,dx \\ &= \ln(x) - \ln(x^{2023}+1)\end{aligned}$$

2 **Integral**: $\int_0^8 \frac{\ln(x)}{\sqrt{x(8-x)}}\,dx$

Solution: Substitute $x = 8\sin^2(u)$ to get the Integral

$$\begin{aligned}\int_0^{\frac{\pi}{2}} 2\ln(8\sin^2(u))\,du &= 3\pi\ln(2) - 4\int_0^{\frac{\pi}{2}} \ln(\sin(u))\,du \\ &= 3\pi\ln(2) - 2\pi\ln(2) \\ &= \pi\ln(2)\end{aligned}$$

where we used the fact that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \ln(\sin(u))\,du + \int_0^{\frac{\pi}{2}} \ln(\sin(u))\,du &= \int_0^{\frac{\pi}{2}} \ln(\sin(u)\cos(u))\,du \\ &= \int_0^{\frac{\pi}{2}} \ln(\frac{\sin(2u)}{2})\,du \\ &= \frac{1}{2}\int_0^{\pi} \ln(\sin(t))\,dt - \frac{\pi}{2}\ln(2) \\ &= \int_0^{\frac{\pi}{2}} \ln(\sin(u))\,du - \frac{\pi}{2}\ln(2).\end{aligned}$$

3 **Integral**: $\int_1^{\infty} \frac{x^4-1}{\ln(x)(x^6+1)}\,dx$

Solution: The idea is to use $\frac{x^4-1}{\log(x)} = \int_0^4 x^t\,dt$ and make substitutions to get the integral into the form of the betafunction. In the end you can use eulers reflection formula and the result is $\ln(2+\sqrt{3})$.

4 **Integral**: $\int_{-1}^{1} \frac{1+x^{2024}}{1+(x^{2024})^{\arctan x}}\,dx$

Solution:

$$\int_{-1}^{1} \frac{1+x^{2024}}{1+(x^{2024})^{\arctan x}}\,dx = \int_{-1}^{1} \frac{1+x^{2024}}{1+(x^{2024})^{-\arctan x}}\,dx$$

So it follows that

$$\begin{aligned}2\int_{-1}^{1} \frac{1+x^{2024}}{1+(x^{2024})^{\arctan x}}\,dx &= \int_{-1}^{1} \frac{1+x^{2024}}{1+(x^{2024})^{\arctan x}}\,dx + \int_{-1}^{1} \frac{1+x^{2024}}{1+(x^{2024})^{-\arctan x}}\,dx \\ &= \int_{-1}^{1} 1+x^{2024} \\ &= 2 + \frac{2}{2025}\end{aligned}$$

and therefore the integral is $1 + \frac{1}{2025}$.

5 **Integral**: $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}(1+x)(x^{2024}+1)}\,dx$

Solution: Substitute $\dfrac{1}{x}$ and $\sqrt{x}$ and use Symmetry to get

$$\begin{aligned}\int_0^\infty \frac{x^{2023}\sqrt{x}}{(1+x)(x^{2024}+1)}\,dx &= 2\int_0^\infty \frac{x^{4048}}{(1+x^2)(x^{4048}+1)}\,dx\\ &= \int_0^\infty \frac{x^{4048}+1}{(1+x^2)(x^{4048}+1)}\,dx\\ &= \frac{\pi}{2}.\end{aligned}$$

6 **Integral** (Tiebreaker): $\displaystyle\int \sqrt{\frac{x}{x+1}}\,dx$

Solution:

$$\begin{aligned}\int 1\cdot\sqrt{\frac{x}{x+1}}\,dx &= (x+1)\sqrt{\frac{x}{x+1}} - \int \frac{1}{2\sqrt{x(x+1)}}\,dx\\ &= \sqrt{x(x+1)} - \int \left(\frac{1}{2\sqrt{x+1}} + \frac{1}{2\sqrt{x}}\right)\frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx\\ &= \sqrt{x(x+1)} - \ln(\sqrt{x}+\sqrt{x+1})\end{aligned}$$

or alternatively subsitute $z = \sqrt{\dfrac{x}{x+1}}$.